

# Fredholm determinant representation for the partition function of the six-vertex model

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## Abstract

The six-vertex model with domain wall boundary conditions is considered. A Fredholm determinant representation for the partition function of the model is obtained. The kernel of the corresponding integral operator depends on Laguerre polynomials.

## 1 Introduction

In the present paper we consider the six-vertex model (ice model) on the two-dimensional square lattice with the special boundary conditions of “domain wall” [1]. First the six-vertex model was solved in the papers [2, 3], where the partition function and bulk free energy were found for the case of periodically boundary conditions. An explicit expression for the partition function with domain wall boundary conditions on the lattice of finite volume was found by A. G. Izergin [4]. Izergin’s representation contains a determinant of  $N \times N$  matrix (where  $N^2$  is the number of vertices of the lattice). Thus, the problem of the thermodynamic limit of the partition function can be reduced to the study of the asymptotic behavior of the determinant at  $N \rightarrow \infty$ .

The goal of this paper is to obtain new representation for the partition function, appropriate for the asymptotic analysis. A Fredholm determinant representation seems to be suitable for this goal. Let us explain in brief the main idea of the method suggested.

Let a matrix  $\mathcal{M}$  of the size  $N$  is given, whose entries  $\mathcal{M}_{jk}(t)$  depend on some parameter  $t$ . Determinants of this matrix  $\det \mathcal{M}$  generate a sequence of functions  $f_N(t)$  for different  $N$ . As a rule the question on the properties of the limiting function  $f_\infty(t) = \lim_{N \rightarrow \infty} f_N(t)$  is rather complicated. In spite of  $\det \mathcal{M}$  is a finite sum of finite products, nevertheless such “explicit” expression for large  $N$  becomes extremely complicated for analysis.

Suppose however that we can replace  $\det \mathcal{M}$  with a Fredholm determinant

$$\det \mathcal{M} = \det(I + V)$$

of an integral operator, whose kernel parametrically depends on  $t$  and  $N$ :  $V(x, y) = V_N(x, y|t)$ . The Fredholm determinant is an infinite sum, and each term of the last one can be presented as a multiple integral, for example,

$$\det(I + V) = \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int V_N(x_1, x_2|t) \cdots V_N(x_k, x_1|t) d^k x \right).$$

While at finite  $N$  the computations of the mentioned integrals are very complicated, at  $N \rightarrow \infty$  the existence of large parameter in the integrands often allows to estimate not only each integral, but the series as a whole. Thus we have the possibility to obtain an estimate for the determinant. Hence, we can say that for  $N \rightarrow \infty$  representations in terms determinants of finite size matrices become “less explicit”, while Fredholm determinants turn to be “more explicit”.

Transformations of finite determinants into Fredholm ones were found to be useful, for instance, in asymptotic analysis of correlation functions of quantum integrable models, asymptotic behavior of orthogonal polynomials, the spectrum of random matrices etc. We hope that in the case under consideration the Fredholm determinant representation will permit eventually to solve the problem of the thermodynamic limit of the partition function of the six-vertex model<sup>1</sup>.

The content of the paper is as follows. In the next section the necessary information on the six-vertex model and Izergin’s formula for the partition function are given. In the third section we transform this formula to the Fredholm determinant representation. In the last section we discuss the outlook for application of the obtained formula to the asymptotic analysis.

## 2 The six-vertex model

Consider a statistical system on a square lattice of the size  $N \times N$ , associating with each edge some variables (classical spin), taking values  $\pm 1$ . Usually these variables are denoted by arrows, identifying  $(\uparrow, \rightarrow)$  with 1, and  $(\downarrow, \leftarrow)$  with  $-1$ . Thus, in general case there are 16 different configurations of arrows in each vertex. The six-vertex is specified by the condition that the number of arrows entering and leaving each vertex coincide. The types of possible vertices are shown on Fig. 1. With each configuration we associate a statistical weights. Generically the last ones can depend on the position of the vertex. We consider the model, which is symmetric with respect to simultaneous reversal of all arrows. Thus, we have  $3N^2$  statistical weights  $w_{jk} = \{a_{jk}, b_{jk}, c_{jk}\}$ , where indices  $j, k = 0, 1, \dots, N - 1$  numerate vertices. The domain wall boundary conditions correspond to the entering arrows on the lower and upper boundaries, and leaving arrows on the left and right ones.

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<sup>1</sup>Recently the thermodynamic limit of the six-vertex model was studied by use of Toda-chain equation [5] and matrix models methods [6]. The approach suggested in the present paper is completely different from the mentioned ones, and we hope that it is also worth attention.

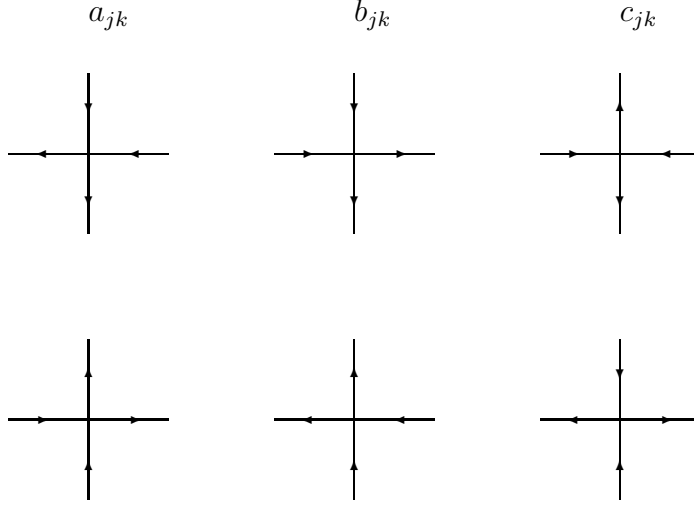


Figure 1: Verteces of the six-vertex model and their statistical weights.

The partition function is given by

$$Z_N = \sum_{\{C\}} \prod_{j,k=0}^{N-1} w_{jk}. \quad (2.1)$$

Here the sum is taken with respect to all possible configurations  $C$  on the whole lattice.

The six-vertex model is exactly solvable, if  $3N^2$  weights  $\{a_{jk}, b_{jk}, c_{jk}\}$  are parameterized by  $2N + 1$  variables  $\lambda_j, \xi_j, (j = 0, 1, \dots, N - 1)$  and  $\eta$ :

$$a_{jk} = \varphi(\lambda_j - \xi_k + \eta), \quad b_{jk} = \varphi(\lambda_j - \xi_k), \quad c_{jk} = \varphi(\eta). \quad (2.2)$$

Parameters  $\lambda_j$  are associated with horizontal lines,  $\xi_j$ —with vertical ones (see. Fig. 2),  $\eta$  is an arbitrary complex constant. Function  $\varphi(x)$  is defined up to normalization factor and is equal to

$$\varphi(x) = \sinh(x), \quad \text{or} \quad \varphi(x) = x. \quad (2.3)$$

In the first case the model is equivalent to the quantum  $XXZ$  magnet, in the second—to the  $XXX$  magnet.

The Izergin's formula for the partition function on the lattice of finite volume with the domain wall boundary conditions has the form

$$Z_N = \frac{\prod_{a,b=0}^{N-1} \varphi(\lambda_a - \xi_b + \eta) \varphi(\lambda_a - \xi_b)}{\prod_{N>a>b\geq 0} \varphi(\lambda_a - \lambda_b) \varphi(\xi_b - \xi_a)} \det \left( \frac{\varphi(\eta)}{\varphi(\lambda_j - \xi_k) \varphi(\lambda_j - \xi_k + \eta)} \right). \quad (2.4)$$

We would like to draw attention of the reader that here and further we numerate indices of matrices starting with zero, but not with one:  $j, k = 0, 1, \dots, N - 1$ .

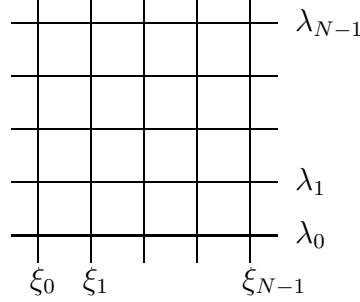


Figure 2: Parametrization of the edges and vertexes of the lattice.

We are interesting in homogeneous limit, when the statistical weights do not depend on the position of the vertex. In other words  $\lambda_j = \lambda$ ,  $\xi_j = \xi$ ,  $j = 0, 1, \dots, N-1$ . The corresponding limit in (2.4) can be obtained via

$$\lim_{\substack{\lambda_j \rightarrow \lambda \\ \xi_k \rightarrow \xi}} \frac{\det[\Phi(\lambda_j, \xi_k)]}{\Delta(\lambda)\Delta(\xi)} = \det \left[ \frac{1}{j!k!} \frac{\partial^j}{\partial \lambda^j} \frac{\partial^k}{\partial \xi^k} \Phi(\lambda, \xi) \right]. \quad (2.5)$$

Here  $\Phi(\lambda, \xi)$  is some enough smooth two-variable function,  $\Delta(\lambda)$  and  $\Delta(\xi)$  are Van der Monde determinants of variables  $\{\lambda\}$  and  $\{\xi\}$  respectively.

In the rational case, when  $\varphi(x) = x$  the direct use of (2.5) gives

$$Z_N^{(XXX)} = \frac{[\nu(\nu + \eta)]^{N^2}}{\prod_{m=0}^{N-1} (m!)^2} \det \left[ (j+k)! \left( \nu^{-j-k-1} - (\nu + \eta)^{-j-k-1} \right) \right], \quad (2.6)$$

where  $\nu = \lambda - \xi$ .

In the trigonometric case it convenient first to use variables  $u_j = e^{2\lambda_j}$ ,  $v_j = e^{2\xi_j}$  and  $q = e^\eta$ . Then (2.4) takes the form

$$Z_N^{(XXZ)} = \prod_{a=0}^{N-1} \left( e^{\lambda_a - \xi_a} \right) \frac{\prod_{a,b=0}^{N-1} \sinh(\lambda_a - \xi_b + \eta)(u_a - v_b)}{\prod_{N>a>b \geq 0} (u_a - u_b)(v_b - v_a)} \det \left( \frac{1}{u_j - v_k} - \frac{q}{qu_j - q^{-1}v_k} \right). \quad (2.7)$$

Now the equation (2.5) can be easily applied, and after simple algebra we obtain

$$Z_N^{(XXZ)} = e^{N\nu} \frac{[\sinh(\nu) \sinh(\nu + \eta)]^{N^2}}{\prod_{m=0}^{N-1} (m!)^2} \det \left[ \frac{(j+k)!}{\sinh^{j+k+1}(\nu)} - \frac{q^{j-k+1}(j+k)!}{\sinh^{j+k+1}(\nu + \eta)} \right]. \quad (2.8)$$

Here as in (2.6)  $\nu = \lambda - \xi$ , and  $q = e^\eta$ .

Below we shall focus on the trigonometric case (2.8), as the most general one. As usual, the answer for the partition function  $Z_N^{(XXX)}$  can be obtained in the limit  $\nu = \epsilon\nu$ ,  $\eta = \epsilon\eta$ ,  $\epsilon \rightarrow 0$ .

Our goal is to transform (2.8) to the expression, suitable for the asymptotic analysis at  $N \rightarrow \infty$ . There are factors with trivial asymptotics in the equation (2.8):  $e^{N\nu}$  and  $[\sinh(\nu) \sinh(\nu + \eta)]^{N^2}$ . It is also easy to evaluate the behavior of the factor  $\prod_{m=0}^{N-1} (m!)^2$  at  $N \rightarrow \infty$ . The main problem is the determinant. In the next section we consider some special transforms of (2.8), which result finally to the new representation for the partition function, containing the determinant of an integral operator.

### 3 Fredholm determinant

As the first step, it is convenient to extract, for example,  $\sinh^{-j-k-1}(\nu)$  out off the determinant (2.8). Then (2.8) becomes

$$Z_N^{(XXZ)} = e^{N\nu} \frac{\sinh^{N^2}(\nu + \eta)}{\prod_{m=0}^{N-1} (m!)^2} \det[A - qtQTATQ^{-1}]. \quad (3.1)$$

Here

$$A_{jk} = (j+k)!, \quad T_{jk} = \delta_{jk} t^j, \quad Q_{jk} = \delta_{jk} q^j, \quad t = \frac{\sinh(\nu)}{\sinh(\nu + \eta)}. \quad (3.2)$$

*Note.* Similarly we could extract terms  $\sinh^{-j-k-1}(\nu + \eta)$ . Then, up to the replacements  $t \rightarrow t^{-1}$ ,  $q \rightarrow q^{-1}$  and trivial common factor, the arising expression coincides with (3.1).

Now we can transform (3.1) as follows:

$$\det[A - qtQTATQ^{-1}] = \det A \cdot \det[I - U], \quad \text{where} \quad U = qtA^{-1}QTATQ^{-1}. \quad (3.3)$$

The advantage of this method is that  $\det A$  cancels the product of factorials in the denominator, since

$$\det A = \prod_{m=0}^N (m!)^2, \quad (3.4)$$

(see Appendix 1). Thus our problem reduces to the derivation of the inverse matrix  $A^{-1}$  and calculation of the product  $U = qtA^{-1}QTATQ^{-1}$ . The inverse matrix  $A^{-1}$  can be easily written in terms of Laguerre polynomials, therefore below we recall their basic properties.

The Laguerre polynomials  $L_n(x)$  are defined by

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \quad (3.5)$$

and generate the orthogonal system on the half-axis  $R_+$  with the weight  $e^{-x}$ :

$$\int_0^\infty L_n(x) L_m(x) e^{-x} dx = \delta_{nm}. \quad (3.6)$$

The key object, determining the matrix  $A^{-1}$ , is the kernel

$$K_n(x, y) = \sum_{k=0}^n L_k(x) L_k(y). \quad (3.7)$$

One more useful representation for  $K_n(x, y)$  follows from Christoffel–Darboux formula:

$$K_n(x, y) = -\frac{n+1}{x-y} (L_{n+1}(x)L_n(y) - L_{n+1}(y)L_n(x)). \quad (3.8)$$

Obviously, for arbitrary polynomial  $\pi_m(x)$  of degree  $m$ , less or equal to  $n$ , holds

$$\int_0^\infty \pi_m(x) K_n(x, y) e^{-x} dx = \pi_m(y), \quad 0 \leq m \leq n. \quad (3.9)$$

To prove (3.9), it is enough to substitute to the l.h.s. the expansion of  $\pi_m(x)$  with respect to Laguerre polynomials  $\pi_m(x) = \sum_{k=0}^m c_k L_k(x)$ , and to use the definition (3.7).

Being a polynomial of variables  $x$  and  $y$ , the kernel  $K_n(x, y)$  can be presented in the form

$$K_n(x, y) = \sum_{j,k=0}^n K_{jk}^{(n)} x^j y^k, \quad (3.10)$$

where

$$K_{jk}^{(n)} = \frac{1}{j!k!} \frac{\partial^{j+k}}{\partial x^j \partial y^k} K_n(x, y) \Big|_{x=y=0}. \quad (3.11)$$

Now we are in position to formulate the theorem on the inverse matrix.

**Theorem 1** *Let  $A_{jk} = (j+k)!$ ,  $j, k = 0, 1, \dots, N-1$ . Then the inverse matrix has the entries*

$$(A^{-1})_{jk} = K_{jk}^{(N-1)}. \quad (3.12)$$

*Proof.* Using the integral representation for  $(j+k)!$ , we have

$$\sum_{l=0}^{N-1} A_{jl} K_{lk}^{(N-1)} = \frac{1}{k!} \frac{\partial^k}{\partial y^k} \sum_{l=0}^{N-1} \frac{1}{l!} \frac{\partial^l}{\partial x^l} \int_0^\infty s^{j+l} e^{-s} K_{N-1}(x, y) ds \Big|_{x=y=0}. \quad (3.13)$$

The sum with respect to  $l$  is the Taylor series (in fact, since  $K_{N-1}(x, y)$  is  $(N-1)$ -degree polynomial of  $x$ , the mentioned series turns to the finite sum). Hence,

$$\sum_{l=0}^{N-1} A_{jl} K_{lk}^{(N-1)} = \frac{1}{k!} \frac{\partial^k}{\partial y^k} \int_0^\infty s^j e^{-s} K_{N-1}(s, y) ds \Big|_{y=0}. \quad (3.14)$$

Due to (3.9) the integral in (3.14) is equal to  $y^j$ . Thus, we arrive at

$$\sum_{l=0}^{N-1} A_{jl} K_{lk}^{(N-1)} = \frac{1}{k!} \frac{\partial^k}{\partial y^k} y^j \Big|_{y=0} = \delta_{jk}. \quad (3.15)$$

The theorem is proved.

*Note.* For generalization of the relationship between Hankel matrices and Christoffel–Darboux kernels see [7].

The obtained explicit expression for  $A^{-1}$  allows us to compute the matrix  $U$  (3.3). Indeed,

$$U_{jk} = \frac{1}{k!} \frac{\partial^k}{\partial y^k} \sum_{l=0}^{N-1} t^{l+k+1} q^{l-k+1} \frac{1}{l!} \frac{\partial^l}{\partial x^l} \int_0^\infty s^{j+l} e^{-s} K_{N-1}(x, y) ds \Big|_{x=y=0}. \quad (3.16)$$

Similarly to the proof of the Theorem the sum with respect to  $l$  easily can be computed and it gives

$$U_{jk} = \frac{qt}{k!} \frac{\partial^k}{\partial y^k} \int_0^\infty x^j (q^{-1}t)^k e^{-x} K_{N-1}(xqt, y) dx \Big|_{y=0}. \quad (3.17)$$

Thus, the partition function  $Z_N^{(XXZ)}$  turns to be proportional to the determinant of the matrix  $\delta_{jk} - U_{jk}$  (3.17). To compute the last one it is convenient to use the equation

$$\det(I - U) = \exp\{\text{tr} \log(I - U)\} = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} \text{tr} U^n\right\}. \quad (3.18)$$

Hence, we need to find the traces of powers of the matrix  $U$ . It is easy to see that

$$(U^n)_{jk} = \frac{(qt)^n}{k!} \frac{\partial^k}{\partial y^k} \int_0^\infty x_1^j (q^{-1}t)^k K_{N-1}(x_1qt, x_2q^{-1}t) \cdots K_{N-1}(x_nqt, y) e^{-\sum_{m=1}^n x_m} d^n x \Big|_{y=0}. \quad (3.19)$$

Indeed, for  $n = 1$  (3.19) coincides with (3.17). Assuming that (3.19) holds for  $U^n$ , we immediately obtain

$$\begin{aligned} \sum_{l=0}^{N-1} (U^n)_{jl} U_{lk} &= \frac{(qt)^{n+1}}{k!} \frac{\partial^k}{\partial y^k} \int_0^\infty x_1^j (q^{-1}t)^k K_{N-1}(x_1qt, x_2q^{-1}t) \cdots \\ &\cdots K_{N-1}(x_nqt, x_{n+1}q^{-1}t) K_{N-1}(x_{n+1}qt, y) e^{-\sum_{m=1}^{n+1} x_m} d^{n+1} x \Big|_{y=0}. \end{aligned} \quad (3.20)$$

(Here the sum with respect to  $l$  again turns to the Taylor series). Hence,

$$\text{tr}(U^n) = (qt)^n \int_0^\infty K_{N-1}(x_1qt, x_2q^{-1}t) \cdots K_{N-1}(x_nqt, x_1q^{-1}t) e^{-\sum_{m=1}^n x_m} d^n x. \quad (3.21)$$

The obtained expression (3.21) for the trace of  $U^n$  allows to replace  $\det(\delta_{jk} - U_{jk})$  with Fredholm determinant. In order to do this, we introduce integral operator  $I - V$ , acting on  $R_+$

$$[(I - V)f](x) = f(x) - \int_0^\infty V(x, y) f(y) dy, \quad (3.22)$$

with the kernel

$$V(x, y) = qt K_{N-1}(xqt, yq^{-1}t) e^{-(x+y)/2} = qt \sum_{k=0}^{N-1} L_k(xqt) L_k(yq^{-1}t) e^{-(x+y)/2}. \quad (3.23)$$

Due to (3.8),  $V(x, y)$  also can be presented in the form

$$V(x, y) = \frac{q^N}{qx - q^{-1}y} \left( L_{N-1}(xqt) L_N(yq^{-1}t) - L_N(xqt) L_{N-1}(yq^{-1}t) \right) e^{-(x+y)/2}. \quad (3.24)$$

Then due to (3.21)

$$\text{tr } U^n = \text{tr } V^n, \quad (3.25)$$

and we finally obtain

$$\det(\delta_{jk} - U_{jk}) = \det(I - V). \quad (3.26)$$

## 4 Discussions

Thus, we have found new representation for the partition function of the six-vertex model, containing the Fredholm determinant of the integral operator with the kernel (3.23), (3.24):

$$Z_N^{(XXZ)} = e^{N\nu} [\sinh(\nu + \eta)]^{N^2} \det(I - V). \quad (4.1)$$

Since the kernel  $V$  is degenerated, we always can turn back to the determinant of the finite size matrix

$$\det \left( I - qt \sum_{k=0}^{N-1} L_k(xqt) L_k(yq^{-1}t) e^{-(x+y)/2} \right) = \det \left( \delta_{jk} - qt \int_0^\infty L_j(xqt) L_k(xq^{-1}t) e^{-x} dx \right). \quad (4.2)$$

However, for the asymptotic analysis at  $N \rightarrow \infty$  the Fredholm determinant representation seems to be preferable due to several reasons.

First, the equation (3.9) shows that the kernel  $K_N(x, y)$  acts as identity operator on the subspace of the polynomials of  $N$ -th degree. Then one can think that for  $N$  going to infinity,  $K_N(x, y)$  goes to identity operator (delta-function). Of course, this limiting procedure requires a serious analysis of convergency. It is remarkable however, that for certain values of  $q$  and  $t$  such a naive interpretation of the kernel (3.23) does holds, and the Fredholm determinant can be evaluated explicitly in the limit  $N \rightarrow \infty$ .

Second, similar integral operators with the kernels, depending on Laguerre polynomials, often appear in the theories of random matrices and random permutations [8, 9]. Some methods of the analysis of such operators can be directly applied to the case under consideration. Besides, at large  $N$  the Laguerre polynomials can be approximated by Bessel functions, what also leads us to the integral operator, analogous to ones, considered in the papers [10, 11, 12].

Third, in particular case of the partition function  $Z_N^{(XXX)}$ , when  $q = 1$ , the kernel  $V(x, y)$  takes the form

$$V(x, y) = \frac{N}{x - y} (L_{N-1}(xt) L_N(yt) - L_N(xt) L_{N-1}(yt)) e^{-(x+y)/2}. \quad (4.3)$$

The integral operator with the kernel (4.3) belongs to the class of integrable integral operators [13]. The corresponding Fredholm determinant turn to be  $\tau$ -function of classical exactly solvable equation and can be evaluated via the matrix Riemann–Hilbert problem methods.



We are planning to present the detailed asymptotic analysis of the partition function of the six-vertex model with domain wall boundary conditions in forthcoming publications.

The author thanks A. R. Its for numerous and useful discussions. The work was supported in parts by RFBR, Grant 99-01-00151 and INTAS-99-1782.

## A Calculation of the determinant

One of the method to proof (3.4) is to use well known formula for Cauchy determinant

$$\frac{\prod_{N>a>b\geq 0} (\alpha_a - \alpha_b)(\beta_b - \beta_a)}{\prod_{a,b=0}^{N-1} (\alpha_a - \beta_b)} = \det \left( \frac{1}{\alpha_j - \beta_k} \right). \quad (\text{A.1})$$

Here  $\{\alpha\}$  and  $\{\beta\}$  are arbitrary complex. Dividing both sides of (A.1) by Van der Monde determinants  $\Delta(\alpha)$  and  $\Delta(\beta)$  and taking homogeneous limit  $\alpha_j \rightarrow \alpha$ ,  $\beta_j \rightarrow \beta$ ,  $j = 0, 1, \dots, N-1$ , we obtain due to (2.5)

$$(\alpha - \beta)^{-N^2} = \prod_{m=0}^{N-1} (m!)^{-2} \det \left( \frac{(j+k)!}{(\alpha - \beta)^{j+k+1}} \right), \quad (\text{A.2})$$

what immediately implies (3.4).

Another method to compute  $\det A$  is based on the application of Laguerre polynomials. Below we present this method, since it allows to find  $A^{-1}$  as well as to prove (3.4).

Using the integral representation for factorial, we have

$$\det A = \det \left( \int_0^\infty e^{-s} s^{j+k} ds \right). \quad (\text{A.3})$$

Denote integration variable in the  $j$ -th row as  $s_j$ . Then we come to the multiple integral

$$\det A = \int_0^\infty ds_0 \cdots ds_{N-1} \det \left( e^{-s_j} s_j^{j+k} \right) = \int_0^\infty ds_0 \cdots ds_{N-1} \prod_{j=0}^{N-1} \left( e^{-s_j} s_j^j \right) \Delta(s), \quad (\text{A.4})$$

where  $\Delta(s)$  is Van der Monde determinant. Now we can use well known identity

$$\Delta(s) = \det[p_k(s_j)], \quad (\text{A.5})$$

where  $p_k(s)$  is a system of *arbitrary* polynomials with the highest coefficient equal to 1:

$$p_k(s) = s^k + o(s). \quad (\text{A.6})$$

In particular we can choose the Laguerre polynomials

$$p_k(s) \equiv \tilde{L}_k(s) = (-1)^k e^s \frac{d^k}{ds^k} (s^k e^{-s}). \quad (\text{A.7})$$

The normalization in (A.7) differs from one in (3.5) by factor  $(-1)^k k!$  in order to provide (A.6). Therefore we denote the polynomials (A.7) as  $\tilde{L}_k$ . The orthogonality condition then takes the form

$$\int_0^\infty \tilde{L}_m(s) \tilde{L}_n(s) e^{-s} ds = (n!)^2 \delta_{nm}. \quad (\text{A.8})$$

Thus, the equation (A.4) becomes

$$\det A = \int_0^\infty ds_0 \cdots ds_{N-1} \prod_{j=0}^{N-1} \left( e^{-s_j} s_j^j \right) \det[\tilde{L}_k(s_j)]. \quad (\text{A.9})$$

Integrating inside the determinant, we obtain

$$\det A = \det \left( \int_0^\infty e^{-s} s^j \tilde{L}_k(s) ds \right). \quad (\text{A.10})$$

Due to orthogonality of the polynomials  $\tilde{L}_k(s)$ , the entries in (A.10) vanish, if  $j < k$ . Thus we deal with the determinant of triangular matrix and, hence,

$$\det A = \prod_{m=0}^{N-1} \left( \int_0^\infty e^{-s} s^m \tilde{L}_m(s) ds \right). \quad (\text{A.11})$$

It remains to observe that expansion of  $s^m$  with respect to  $\tilde{L}_m(s)$  has the form

$$s^m = \tilde{L}_m(s) + \text{polynomials of lower degree}$$

therefore due to (A.8)

$$\det A = \prod_{m=0}^{N-1} (m!)^2, \quad (\text{A.12})$$

which ends the proof.

The described method allows to compute not only  $\det A$ , but the minors of the size  $N - 1$  as well. This provides the possibility to find  $A^{-1}$ .

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